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On complexity of minimization and compression problems for models of sequential choice ☆

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Abstract

A model of sequential choice of depth k for binary relations $r_1, r_2, \ldots, r_k$ on a set A of alternatives relates each $X \subseteq A$ to its subset $C_{r_k}(\ldots C_{r_2}(C_{r_1}(X))\ldots)$, where $C_r(Y)=\{y \in Y \mid (\forall z \in Y)\, z\bar{r}y\}$, $Y \subseteq A$. The minimization problem is of building an equivalent model of minimal depth; the compression problem is posed similarly, but the model built must satisfy some "insertion" condition. We prove that for $k \geqslant 3$, the minimization and compression problems for models of depth k are NP-hard (for $k = 2$, they are polynomial). Parameters of local algorithms solving these problems are investigated, and it is shown that the compression problem is decidable by algorithms working with neighbourhoods of size 3, whereas the minimization problem is not decidable for any finite neighbourhood size. For an arbitrary k, a model of depth k is built such that its minimization problem is not decidable with the use of neighbourhoods smaller than the whole model. © 2002 Elsevier B.V. All rights reserved.

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1. Introduction

Wide usage of computers in decision procedures gives rise to studying the formal models of choice. A choice model M on the set A of alternatives assigns each set $X \subseteq A$ the subset $C_M(X) \subseteq X$ of alternatives chosen and thus generates a choice function $C_M : 2^A \to 2^A$. Let M belong to a class $\mathscr{M}$ of models, and let a quantitative parameter of complexity be defined for models from $\mathscr{M}$. Given a model M, the minimization problem is to find the simplest model from $\mathscr{M}$ representing the function C_M. Note that

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the simpler model is, the more accurate it predicts the choice in new situations (for pattern recognition, a similar fact is verified in [7]; its analogue for choice problems can be found in [6]).

The basic model in the choice theory is the binary relation choice model. Let r be a (binary) relation on the set A of alternatives. For $x, y \in A$, the relation xry is interpreted as "the possibility x is better than y", and the set $C_r(X)$ contains exactly all best alternatives in X with respect to r (i.e., $x \in X$ such that no y in X satisfies yrx). More complicated models can be built from the basic model. One of them is the model of sequential choice of depth k. It is defined by the set $(r_1, r_2, \ldots, r_k)$ of binary relations on A; the choice from the set of alternatives X, $X \subseteq A$, takes k stages. First we choose the alternatives of X which are best with respect to r_1, then we choose from them the alternatives best with respect to r_2, and so on; at last, we choose the alternatives best with respect to r_k. Along with the choice function $C_{\mathbf{r}}$ generated by the model $\mathbf{r} = r_1 r_2 \ldots r_k$, we can introduce auxiliary functions $C_{r_1 \ldots r_i}$ of choice after first i stages, $i \leqslant k$. The minimization and compression problems are posed for $\mathbf{r}$. The first problem is to find an equivalent (i.e., representing the same function $C_{\mathbf{r}}$) model of minimal depth; the second one is to find an equivalent model of minimal depth which does not generate auxiliary choice functions new with respect to $\mathbf{r}$.

These problems can be stated in terms of graph theory. To each relation r, we assign a directed graph G on the vertex set A such that $(x, y) \in G \Leftrightarrow xry$. The set $C_G(X)$ of vertices chosen from $X \subseteq A$ in G consists of all source (i.e., not having incoming edges) vertices of the subgraph G_X induced by X. The sequential choice model considered as a set of graphs $\mathbf{G} = G_1 G_2 \ldots G_k$ represents the choice function $C_{\mathbf{G}}(X) = C_{G_k}(\ldots C_{G_2}(C_{G_1}(X)) \ldots)$. The minimization problem is to find the shortest sequence $\mathbf{G}'$ of graphs such that $C_{\mathbf{G}} = C_{\mathbf{G}'}$. The compression problem can be stated analogously.

Most of previous results considered the choice model of depth 2. In this case, the minimization and compression problems coincide and can be reduced to the problem of representability of the function $C_{r_1 r_2}$ by one relation. In [2,5] (see also [1,3]) this problem is solved for the case when r_1 and r_2 are linear or partial orders (the statement of the problem solved differs somewhat from ours). Some special cases for the case of $r_1 \subseteq r_2$ are studied in [8,9]. An effective (polynomial) algorithm solving the minimization problem for the general choice model of depth 2 is offered in [6]. For models of arbitrary depth k, an algorithm is described in [6] which solves the minimization problem almost always (but not always) for k=const and increasing number n of alternatives. In the same monograph, an asymptotically tight bound for depth is found as $n \to \infty$.

In this paper, we show that the minimization and compression problems for models of depth k are NP-hard for $k \geqslant 3$ (see e.g. [4]). Since the minimization and the compression problems can be polynomially reduced to each other, their time complexity behaves similarly.

A significant difference between the two problems arises when we study their informational parameters. We use the approach by Zhuravlev [10,11]; i.e., characterize the problem by the least number d such that the problem can be solved by a local algorithm of index d, which is an algorithm dealing with neighbourhoods of size d (more information about neighbourhoods can be found in [12]). The model introduced in this paper is different from the model of a local algorithm from [10,11] built for computing predicates and assuming the system of neighbourhoods to be fixed. The minimization and compression problems we consider require building objects, and during the work of the algorithm, neighbourhoods change according to transformations of model $\mathbf{r}$. To emphasize the difference between models, we use the term "local procedure" instead of "local algorithm".

We show that the compression problem for sequential choice models can be solved by local procedures of index 3, and the minimization problem is not decidable for any finite d. For each k and d we build a model whose depth k cannot be lowered by local procedures of index d but can be reduced to d by procedures of index $d+1$ (increasing k, we can gain arbitrarily much). For $d = k - 1$, this construction gives a model of depth k which cannot be simplified by procedures dealing with neighbourhoods distinct from the whole model.

As corollaries, we obtain a series of results concerning sequential choice models. A polynomial algorithm is found that solves the model compression problem for the case of acyclic relations (recall that for arbitrary relations, the problem is NP-hard). The NP-hardness of the model equivalence problem and non-existence of a finite complete system of equivalent transformations are proved.

2. The problems

Given a set of objects (alternatives) A, a map $C: 2^A \to 2^A$ is called a *choice function* if $C(X) \subseteq X$ for each $X \in 2^A$. Objects of $C(X)$ are said to be *chosen* from X.

Let r be an irreflexive (i.e. satisfying $x\bar{r}x$, $x \in A$) binary relation on A. It is assigned the choice function

$$C_r(X) = \{x \in X \mid (\forall y \in X) y\bar{r}x\}.$$

A model of *sequential choice* is defined by a set of (binary irreflexive) relations $(r_1, r_2, \ldots, r_k)$. A model $\mathbf{r} = r_1 r_2 \ldots r_k$ realizes the choice function

$$C_{\mathbf{r}}(X) = C_{r_1 r_2 \ldots r_k}(X) = C_{r_k}(\ldots C_{r_2}(C_{r_1}(X)) \ldots).$$

The number $k = k(\mathbf{r})$ is called the *depth* of $\mathbf{r}$. In what follows, we shall imply models of sequential choice while speaking about models.

A model $\mathbf{r}$ is equipped with *truncated models* $\mathbf{r}|_i = r_1 \ldots r_i$, $1 \leqslant i \leqslant k$. Let $\mathscr{C}_{\mathbf{r}} = \{C_{\mathbf{r}|_1}, \ldots, C_{\mathbf{r}|_k}\}$ be the set of functions realized by truncated models. Clearly, we have $C_{\mathbf{r}|_1} \supseteq C_{\mathbf{r}|_2} \supseteq \cdots \supseteq C_{\mathbf{r}|_k}$, where the notation $C' \supseteq C$ for choice functions C' and C means that $C'(X) \supseteq C(X)$ for all $X \subseteq A$. We shall say that the model $\hat{\mathbf{r}}$ is

- *equivalent* to a model $\mathbf{r}$ if $C_{\hat{\mathbf{r}}} = C_{\mathbf{r}}$;
- *indistinguishable* from $\mathbf{r}$ if $k(\hat{\mathbf{r}}) = k(\mathbf{r})$ and $C_{\hat{\mathbf{r}}|_i} = C_{\mathbf{r}|_i}$, $1 \leqslant i \leqslant k(\mathbf{r})$;
- *embedded* to $\mathbf{r}$ if $\mathscr{C}_{\hat{\mathbf{r}}} \subseteq \mathscr{C}_{\mathbf{r}}$.

A model $\hat{\mathbf{r}}$ is called *minimal* (*shortest*) for $\mathbf{r}$ if it is equivalent to $\mathbf{r}$ (equivalent and embedded to $\mathbf{r}$) and has minimal possible depth. The problem of building the minimal (shortest) model is called the problem of *minimization* (respectively, *compression*) of the model. This paper is devoted to the complexity analysis of the problems stated.

3. Equivalent transformations of models

First, let us consider transformations of indistinguishable models. To a model $\mathbf{r} = r_1 \dots r_k$, we assign the *lower relation set* $\mathbf{r}^- = (r_1^-, \dots, r_k^-)$ and the *upper relation set* $\mathbf{r}^+ = (r_1^+, \dots, r_k^+)$ by putting

$$r_i^- = r_i \Big\backslash \bigcup_{1 \leqslant j \leqslant i-1} (r_j \cup r_j^{-1}), \quad r_i^+ = r_i \cup \bigcup_{1 \leqslant j \leqslant i-1} (r_j \cup r_j^{-1}), \quad 1 \leqslant i \leqslant k.$$

We shall say that elements x are y *linked* by the relation r if xry or yrx. In these terms, the relation r_i^- is built by deletion from r_i all pairs (x, y) such that x and y are linked by one of the previous relations $r_1, \dots, r_{i-1}$.

Lemma 1. *The model* $\hat{\mathbf{r}} = \hat{r}_1 \dots \hat{r}_k$ *is indistinguishable from the model* $\mathbf{r}$ *if and only if* $r_i^- \subseteq \hat{r}_i \subseteq r_i^+$, $1 \leqslant i \leqslant k$.

Proof. (a) First establish the indistinguishability of the models $\mathbf{r}$ and $\mathbf{r}^-$. Using induction on i, $1 \leqslant i \leqslant k$, we shall prove that $C_{\mathbf{r}|_i}(X) = C_{\mathbf{r}^-|_i}(X)$ for every $X \subseteq A$. The equality $C_{\mathbf{r}|_1}(X) = C_{\mathbf{r}^-|_1}(X)$ follows from $r_1 = r_1^-$. Let $C_{\mathbf{r}|_{i-1}}(X) = C_{\mathbf{r}^-|_{i-1}}(X)$. If the elements x and y are linked by a relation r_j, $1 \leqslant j \leqslant i-1$, then there is at most one of them in the set $X_{i-1} = C_{\mathbf{r}_{i-1}}(X)$. So, the link between x and y in r_i has no effect on the choice of $C_{r_i}(X_{i-1})$, and r_i^- can be used instead of r_i. Thus,

$$C_{\mathbf{r}|_i}(X) = C_{r_i}(C_{\mathbf{r}|_{i-1}}(X)) = C_{r_i^-}(C_{\mathbf{r}|_{i-1}}(X)) = C_{r_i^-}(C_{\mathbf{r}^-|_{i-1}}(X)) = C_{\mathbf{r}^-|_i}(X).$$

(b) Now let us verify that the models $\mathbf{r}$ and $\hat{\mathbf{r}}$ are indistinguishable if and only if $\mathbf{r}^- = \hat{\mathbf{r}}^-$. If $\mathbf{r}^- = \hat{\mathbf{r}}^-$, then $\mathbf{r}$ and $\hat{\mathbf{r}}$ are indistinguishable because they are indistinguishable from $\mathbf{r}^-$ and $\hat{\mathbf{r}}^-$ due to (a). Suppose that $\mathbf{r}^- \neq \hat{\mathbf{r}}^-$. Let the sets $\mathbf{r}^-$ and $\hat{\mathbf{r}}^-$ first differ in the ith relation, and let for definiteness sake $(x, y) \in r_i^- \setminus \hat{r}_i^-$. Then x and y are not linked in $r_1^-, \dots, r_i^-$ and thus in $\hat{r}_1^-, \dots, \hat{r}_{i-1}^-$. Putting $X = \{x, y\}$, we obtain that $y \notin C_{r_i^-}(\{x, y\}) = C_{\mathbf{r}^-|_i}(\{x, y\})$ and $y \in C_{\hat{r}_i^-}(\{x, y\}) = C_{\hat{\mathbf{r}}^-|_i}(\{x, y\})$. It means that $\mathbf{r}^-$ and $\hat{\mathbf{r}}^-$ are distinguishable. Due to (a), the same can be said about $\mathbf{r}$ and $\hat{\mathbf{r}}$.

The statement of the lemma follows from (b) and the easy fact that the inclusions $r_i^- \subseteq \hat{r}_i \subseteq r_i^+$, $1 \leqslant i \leqslant k$, are necessary and sufficient for the equality between $\mathbf{r}^-$ and $\hat{\mathbf{r}}^-$. $\square$

In what follows, two special kinds of models will be important. Relations r and $\hat{r}$ will be called *separated* if $(r \cup r^{-1}) \cap (\hat{r} \cup \hat{r}^{-1}) = \emptyset$ (i.e., elements linked by one of them are not linked by the other). A sequential choice model $\mathbf{r}$ will be called *reduced* if the relations of $\mathbf{r}$ are mutually separated. Let $r|_{x,y} = r \cap \{(x, y), (y, x)\}$ be the contraction of a relation r to the set $\{x, y\}$. A model $\mathbf{r} = r_1 \dots r_k$ will be called *canonical* if $r_i|_{x,y} \neq \emptyset \Rightarrow r_{i+1}|_{x,y} = r_i|_{x,y}$, $1 \leqslant i \leqslant k-1$, $x, y \in A$ (i.e., if some x and y are linked by a relation, they are similarly linked by all subsequent relations).

Lemma 2. *For each model* $\mathbf{r}$, *there exist a unique indistinguishable reduced model* $\mathbf{r}^0$ *and a unique indistinguishable canonical model* $\mathbf{r}^1$. *They are defined by formulas* $\mathbf{r}^0 = \mathbf{r}^-$, $\mathbf{r}^1 = (r_1^-, r_1^- \cup r_2^-, \ldots, r_1^- \cup \cdots \cup r_k^-)$.

Proof. According to Part (a) in the proof of Lemma 1, the model $\mathbf{r}^-$ is indistinguishable from $\mathbf{r}$. Clearly, $\mathbf{r}^-$ is reduced. Let $\hat{\mathbf{r}}$ be a reduced model of depth k distinct from $\mathbf{r}^-$. Then $\hat{\mathbf{r}}^- = \hat{\mathbf{r}} \neq \mathbf{r}^-$, and according to (b) of Lemma 1, the models $\hat{\mathbf{r}}$ and $\mathbf{r}$ are indistinguishable.

It can be easily seen that the model $\mathbf{r}^1$ satisfies the conditions $r_i^- \subseteq r_i^1 \subseteq r_i^+$, $1 \leqslant i \leqslant k$, and due to Lemma 1, it is indistinguishable from $\mathbf{r}$. Clearly, $\mathbf{r}^1$ is canonical. Consider a canonical model $\hat{\mathbf{r}}$ indistinguishable from $\mathbf{r}$. We see that $\hat{\mathbf{r}}^- = (\hat{r}_1, \hat{r}_2 \setminus \hat{r}_1, \ldots, \hat{r}_k \setminus \hat{r}_{k-1})$. It follows from Part (b) of Lemma 1 that $\hat{\mathbf{r}}^- = \mathbf{r}^-$. Consequently, $\hat{\mathbf{r}} = (r_1^-, r_1^- \cup r_2^-, \ldots, r_1^- \cup \cdots \cup r_k^-) = \mathbf{r}^1$. □

Let us introduce the *lexicographical* operation $\otimes$ on the set of relations by putting $x(r_1 \otimes r_2)y \Leftrightarrow xr_1y \vee y\bar{r}_1x \wedge xr_2y$. It can be easily checked that $(r_1 \otimes r_2) \otimes r_3 = r_1 \otimes (r_2 \otimes r_3)$. Thus, we may consider a polyadic operation $r_1 \otimes r_2 \otimes \cdots \otimes r_k$ defined by $\otimes\mathbf{r}$, where $\mathbf{r} = r_1 \ldots r_k$. Note that if the set $\mathbf{r}$ is reduced, then $\otimes\mathbf{r}$ coincides with $\cup\mathbf{r}$, where $\cup\mathbf{r} = r_1 \cup r_2 \cup \cdots \cup r_k$. Clearly, the canonical model for $\mathbf{r}$ can be represented as

$$\mathbf{r}^1 = (r_1, r_1 \otimes r_2, \ldots, r_1 \otimes r_2 \otimes \cdots \otimes r_k) = (\mathbf{r}|_1, \otimes(\mathbf{r}|_2), \ldots, \otimes(\mathbf{r}|_k)).$$

Given a choice function C, its *lower approximation* (in the class of relations) is defined to be the choice function C_r realized by some relation r such that $C_r \subseteq C$ and for each relation r' satisfying $C_{r'} \subseteq C$ satisfies $C_{r'} \subseteq C_r$. Clearly, if the lower approximation exists, it is unique and it uniquely determines the relation $r : xry \Leftrightarrow y \notin C_r(\{x, y\})$.

Lemma 3. *For each model* $\mathbf{r} = r_1 \ldots r_k$, *the lower approximation of the function* $C_\mathbf{r}$ *exists and is realized by the relation* $\otimes\mathbf{r}$.

Proof. Let $\mathbf{r}^1 = r_1^1 \ldots r_k^1$ be the corresponding canonical model. Then we have $C_{\mathbf{r}^1} = C_\mathbf{r}$, $r_1^1 \subseteq \cdots \subseteq r_k^1$, and $r_k^1 = \otimes\mathbf{r}$. Consider an arbitrary $X \subseteq A$. If $x \in X \setminus C_\mathbf{r}(X)$, then there exist some $y \in X$ and a relation r_i^1 from $\mathbf{r}^1$ such that $(y, x) \in r_i^1$. Since $r_i^1 \subseteq \otimes\mathbf{r}$, we have $(y, x) \in \otimes\,\mathbf{r}$, and thus $x \notin C_{\otimes\mathbf{r}}(X)$. It means that $C_{\otimes\mathbf{r}}(X) \subseteq C_\mathbf{r}(X)$.

Let r' be a relation satisfying $C_{r'} \subseteq C_\mathbf{r}$, and (x, y) be a pair from $\otimes\mathbf{r} = r_k^1$. We take the relation r_j^1 from $\mathbf{r}^1$ in which x and y are linked for the first time. Since the model is canonical, we have $(x, y) \in r_j^1$ and thus $y \notin C_{r_j^1}(\{x, y\}) = C_{\mathbf{r}^1}(\{x, y\}) = C_\mathbf{r}(\{x, y\})$. But then $y \notin C_{r'}(\{x, y\})$ and thus $(x, y) \in r'$. It means that $r' \supseteq \otimes\mathbf{r}$ and $C_{r'} \subseteq C_{\otimes\mathbf{r}}$. □

Corollary 1. *If the models* $\mathbf{r} = r_1 \ldots r_k$ *and* $\mathbf{r}' = r_1' \ldots r_l'$ *are equivalent, then* $\otimes\mathbf{r} = \otimes\mathbf{r}'$. *In particular, if the models* $\mathbf{r}$ *and* $\mathbf{r}'$ *are reduced, then* $\cup\mathbf{r} = \cup\mathbf{r}'$, *and if they are canonical, then* $r_k = r_l'$.

This corollary follows from Lemma 3 and the uniqueness of the lower approximation.

Corollary 2. *If the choice function $C_{\mathbf{r}}$, $\mathbf{r}=r_1\ldots r_k$, can be represented by a relation r, then $r=\otimes\mathbf{r}$ ($r=\cup\mathbf{r}$ for the reduced model and $r=r_k$ for the canonical one).*

This corollary follows from the fact that if the choice function C can be represented by a relation, then its lower approximation coincides with C.

Let us denote by $\{\mathbf{r}\}$ the set of relations $\{r_1,\ldots,r_k\}$ appearing in the model $\mathbf{r}=r_1\ldots r_k$.

Lemma 4. *If $\mathbf{r}$ and $\mathbf{r}'$ are canonical models, and $\mathbf{r}'$ is embedded to $\mathbf{r}$, then $\{\mathbf{r}'\}\subseteq\{\mathbf{r}\}$.*

Proof. Let $\mathbf{r}=r_1\ldots r_k$ and $\mathbf{r}'=r'_1\ldots r'_{k'}$. By the definition of embedded model, $\mathscr{C}_{\mathbf{r}'}\subseteq\mathscr{C}_{\mathbf{r}}$, and thus for each i, $1\leqslant i\leqslant k'$, there exists some j, $1\leqslant j\leqslant k$, such that $C_{\mathbf{r}'|_i}=C_{\mathbf{r}|_j}$. These functions have equal lower approximations, and due to Lemma 3 (since $\mathbf{r}'|_i$ and $\mathbf{r}|_j$ are canonical models) they are realized by the relations r'_i and r_j. This implies $r'_i=r_j$. □

Note that Lemma 4 holds only for canonical models, and that the condition $\{\mathbf{r}'\}\subseteq\{\mathbf{r}\}$ is not sufficient for models to be embedded.

Let us introduce the *superposition* operation $C_1\circ C_2$ of the choice functions C_1 and C_2 by putting $(C_1\circ C_2)(X)=C_2(C_1(X))$.

Lemma 5. *The superposition operation is associative, i.e.,*

$$(C_1\circ C_2)\circ C_3=C_1\circ(C_2\circ C_3).$$

Indeed, each choice function is a map $2^A\to 2^A$, and a superposition corresponds to a product of maps which is known to be associative.

Due to Lemma 5, we may consider a polyadic operation $C_1\circ C_2\circ\cdots\circ C_k$ where parentheses can be placed arbitrarily. Then the function of sequential choice can be written down as $C_{\mathbf{r}}=C_{r_1}\circ C_{r_2}\circ\cdots\circ C_{r_k}$. A fragment $r_ir_{i+1}\ldots r_j$, $1\leqslant i\leqslant j\leqslant k$, of a model $\mathbf{r}=r_1\ldots r_k$ will be denoted by $\mathbf{r}|_j^i$, and the numbers i and j will be called its (left and right) *boundaries*. Since $\mathbf{r}=(\mathbf{r}|_i^1,\mathbf{r}|_j^{i+1},\mathbf{r}|_k^{j+1})$, and the superposition is associative, we have

$$C_{\mathbf{r}}=C_{\mathbf{r}|_i^1}\circ C_{\mathbf{r}|_j^{i+1}}\circ C_{\mathbf{r}|_k^{j+1}}. \tag{1}$$

Let sequences of relations $\mathbf{f}'=r'_1\ldots r'_u$ and $\mathbf{f}''=r''_1\ldots r''_v$ be given, and let the model $\mathbf{r}$ contain the fragment $\mathbf{f}'$, i.e., $\mathbf{r}=\mathbf{r}_1\mathbf{f}'\mathbf{r}_2$ for some (possibly empty) $\mathbf{r}_1$ and $\mathbf{r}_2$. We shall say that the *local transform* $\mathbf{f}'\to\mathbf{f}''$ is applied to $\mathbf{r}$ if the fragment $\mathbf{f}'$ in $\mathbf{r}$ is replaced by $\mathbf{f}''$. The result is the model $\hat{\mathbf{r}}=\mathbf{r}_1\mathbf{f}''\mathbf{r}_2$. A local transform $\mathbf{f}'\to\mathbf{f}''$ will be called *correct* if it maps each model (for which it is applicable) to an equivalent one.

Lemma 6. *A local transform $\mathbf{f}'\to\mathbf{f}''$ is correct if and only if $C_{\mathbf{f}'}=C_{\mathbf{f}''}$.*

Proof. If $C_{\mathbf{f}'}=C_{\mathbf{f}''}$, then using the definitions above and (1), we obtain

$$C_{\hat{\mathbf{r}}}=C_{\mathbf{r}_1}\circ C_{\mathbf{f}''}\circ C_{\mathbf{r}_2}=C_{\mathbf{r}_1}\circ C_{\mathbf{f}'}\circ C_{\mathbf{r}_2}=C_{\mathbf{r}}.$$

In the case of $C_{\mathbf{f}'} \neq C_{\mathbf{f}''}$, the application of the transform $\mathbf{f}' \to \mathbf{f}''$ to the model $\mathbf{r} = \mathbf{f}'$ gives a non-equivalent model $\mathbf{f}''$. $\square$

4. NP-hard and polynomial cases: the analysis

Recall some well-known notions. The *product* of relations r_1 and r_2 is the relation $r_1 \cdot r_2$ such that $x(r_1 \cdot r_2)y \Leftrightarrow \exists z(xr_1z \wedge zr_2y)$. A relation r is called *acyclic* if it contains no *cycles* of the form $x_1rx_2 \wedge x_2rx_3 \wedge \cdots \wedge x_{s-1}rx_s \wedge x_srx_1$, $s \geqslant 1$. For $x \in A$, $Y \subseteq A$, and a relation r we put $r^{-1}(x) = \{y \mid yrx\}$ and $r|_Y = r \cap Y^2$ (the contraction of r to Y).

Lemma 7. *Let* $\mathbf{r} = r_1r_2$ *be a reduced model of depth* 2. *The function* $C_{\mathbf{r}}$ *can be represented by a relation if and only if*

$$r_1 \cdot r_2 \subseteq r_1 \cup r_2 \tag{2}$$

and for each $x \in A$ *the relation* $r_1|_{r_2^{-1}(x)}$ (*the contraction of* r_1 *to* $r_2^{-1}(x)$) *is acyclic.*

Proof. *Necessity*: If the choice function $C_{\mathbf{r}}$ can be represented by a relation r, then due to Corollary 2 we have $r = r_1 \otimes r_2 = r_1 \cup r_2$.

Suppose that (2) does not hold and x, y, z are such that xr_1y, yr_2z, and $x\bar{r}z$. Put $X = \{x, y, z\}$. It follows from $(y,z) \in r_2 \subseteq r$ that $z \notin C_r(X)$. On the other hand, $x\bar{r}z$ implies $x\bar{r}_1z$, while yr_2z and the fact that r_1 and r_2 are separated imply $y\bar{r}_1z$. Thus $z \in C_{r_1}(X)$. Due to the fact that $y \notin C_{r_1}(X)$ since xr_1y and $x\bar{r}_2z$ (the latter relation holds because $r_2 \subseteq r$ and $x\bar{r}z$), we obtain $z \in C_{r_2}(C_{r_1}(X)) = C_{\mathbf{r}}(X)$. It means that $C_{\mathbf{r}} \neq C_r$, and the choice function $C_{\mathbf{r}}$ cannot be represented by a relation.

Suppose that for some x, the relation $r_1|_{r_2^{-1}(x)}$ contains a cycle, i.e., there exist $x_1, \ldots, x_s \in r_2^{-1}(x)$ such that $x_1r_1x_2 \wedge \cdots \wedge x_{s-1}r_1x_s \wedge x_sr_1x_1$. Put $X = \{x, x_1, \ldots, x_s\}$. Since x is linked with $x_1, \ldots, x_s$ in r_2, and the relations r_1 and r_2 are separated, there are no pairs $(x_1,x), \ldots, (x_s,x)$ in r_1, and consequently $x \in C_{r_1}(X)$. At the same time, clearly, $x_1, \ldots, x_s \notin C_{r_1}(X)$, and thus $C_{\mathbf{r}}(X) = C_{r_1}(X) = \{x\}$. But $(x_1,x) \in r_1 \cup r_2 = r$ implies $x \notin C_r(X)$. This gives $C_{\mathbf{r}} \neq C_r$ and means the non-representability of the function $C_{\mathbf{r}}$ by a relation.

Sufficiency: Let a model $\mathbf{r}$ satisfy the conditions of the lemma. Let us prove that $C_{\mathbf{r}} = C_r$, where $r = r_1 \cup r_2$. Due to Lemma 3, the choice function C_r is a lower approximation. So, we must only check that $C_{\mathbf{r}} \subseteq C_r$.

Consider a set X and its element $x \in X$ such that $x \notin C_r(X)$. Suppose that $x \in C_{\mathbf{r}}(X)$. Since $x \notin C_r(X)$, there exists an element x_1 in X such that x_1rx. Here the condition $x \in C_{\mathbf{r}}(X)$ can be satisfied only if x_1r_2x and there exists an $x_2 \in X$ such that $x_2r_1x_1$. It follows from $x_2r_1x_1$, x_1r_2x, and (2) that x_2rx. Replacing x_1 by x_2 in the arguments above, we obtain x_2r_2x and see that an element x_3 exists such that x_3r_1x, and so on. Since the set A is finite, the sequence $x_1, x_2, x_3, \ldots$ has repeated entries. If $x_u = x_v$, $u < v$, then there exists a cycle $x_ur_1x_{u+1}r_1 \ldots r_1x_{v-1}r_1x_u$ in r_1 such that $x_u, \ldots, x_{v-1} \in r_2^{-1}(x)$. This contradicts the condition that $r_1|_{r_2^{-1}(x)}$ is an acyclic relation. $\square$

Lemma 8. *Let* $\mathbf{r}=r_1r_2r_3$ *be a reduced model of depth* 3, *then the function* $C_{\mathbf{r}}$ *can be represented by a model* $\mathbf{r}'=r_1r$ *for some* r *if and only if*

$$r_2\cdot r_3 \subseteq r_1\cup r_1^{-1}\cup r_2\cup r_3 \tag{3}$$

and for each $x\in A$ *every cycle of the relation* $r_2|_{r_3^{-1}(x)}$ (*i.e., of the contraction of* r_2 *to* $r_3^{-1}(x)$)) *passes through some pair of elements linked in* r_1.

Proof. *Necessity*: The model $\mathbf{r}'=r_1r$ can be assumed to be reduced. If $\mathbf{r}'$ is equivalent to $\mathbf{r}$, then due to Corollary 1 we have $r_1\cup r_2\cup r_3=r_1\cup r$ and $r=r_2\cup r_3$ since the models are reduced.

Suppose that (3) does not hold, and for some x,y,z we have xr_1y, yr_2z, and $x(r_1\cup r_1^{-1}\cup r_2\cup r_3)z$. Then x and z are not linked in r_1, and $x\bar{r}z$. Put $X=\{x,y,z\}$. As in the proof of Lemma 7, we can check that $C_{r_3}(C_{r_2}(X))\neq C_r(X)$. Since $\mathbf{r}$ is reduced, x is not linked with y, and y with z in r_1. Thus, since x and z are also not linked, we have $C_{r_1}(X)=X$. Consequently,

$$\begin{aligned} C_{\mathbf{r}}(X) &= C_{r_3}(C_{r_2}(C_{r_1}(X)))=C_{r_3}(C_{r_2}(X)) \\ &\neq C_r(X)=C_r(C_{r_1}(X))=C_{\mathbf{r}'}(X), \end{aligned} \tag{4}$$

i.e., $\mathbf{r}$ and $\mathbf{r}'$ are not equivalent.

Suppose that for some x, the relation $r_2|_{r_3^{-1}(x)}$ contains a cycle whose elements are not mutually linked in r_1, i.e., there exist $x_1,\dots,x_s$ in $r_3^{-1}(x)$ such that $x_1r_2x_2\wedge\cdots\wedge x_{s-1}r_2x_s\wedge x_sr_2x_1$ and $x_i\bar{r}_1x_j$, $1\leqslant i,j\leqslant s$. Since $\mathbf{r}$ is reduced and $xr_3x_1,\dots,xr_3x_s$, it follows that x is not linked with $x_1,\dots,x_s$ in r_1. Put $X=\{x,x_1,\dots,x_s\}$. Then $C_{r_1}(X)=X$. Like in the proof of Lemma 7, we can check that $C_{r_3}(C_{r_2}(X))\neq C_r(X)$. After this, the fact that the models $\mathbf{r}$ and $\mathbf{r}'$ are not equivalent follows from (4).

Sufficiency: Let the conditions of Lemma 8 be satisfied. Consider an arbitrary $X\subseteq A$ and put $Y=C_{r_1}(X)$, $\hat{r}_2=r_2|_Y$, $\hat{r}_3=r_3|_Y$, $\hat{r}=r|_Y$. For each x, the relation $\hat{r}_2|_{\hat{r}_3^{-1}(x)}$ is acyclic: otherwise $r_2|_{r_3^{-1}(x)}$ contains a cycle whose elements are mutually not linked in r_1 (the set Y does not contain elements linked in r_1). The condition $\hat{r}_2\cdot\hat{r}_3\subseteq\hat{r}_2\cup\hat{r}_3$ follows from (3) and the fact that the elements of Y are not linked in r_1. Applying Lemma 7 to $\hat{r}_2$ and $\hat{r}_3$, we conclude that $C_{\hat{r}_3}(C_{\hat{r}_2}(Z))=C_{\hat{r}}(Z)$ for each $Z\subseteq A$. Thus,

$$\begin{aligned} C_{\mathbf{r}}(X) &= C_{r_3}(C_{r_2}(C_{r_1}(X)))=C_{r_3}(C_{r_2}(Y))=C_{r_3|_Y}(C_{r_2|_Y}(Y)) \\ &= C_{\hat{r}_3}(C_{\hat{r}_2}(Y))=C_{\hat{r}}(Y)=C_{r|_Y}(Y)=C_r(Y)=C_r(C_{r_1}(X))=C_{\mathbf{r}'}(X). \quad\square \end{aligned}$$

Let us state the main claim of this section. The notions of NP-completeness, NP-hardness and polynomial solvability are assumed to be known (see e.g. [4]).

Theorem 1. (a) *For all* $k\geqslant 3$, *the compression and minimization problems for models of sequential choice of depth* k *are NP-hard.*

(b) *For* $k=2$, *these problems are polynomial.*

Proof. (a) First, consider the case of $k=3$. We shall prove the NP-hardness of the compression and minimization problems by reducing to them the problem PATH WITH

FORBIDDEN PAIRS (problem GT 54 from [4]). We shall use the following special case of this problem (see Remark to GT 54). Given a acyclic digraph $G \subseteq V^2$, a set $U = \{(a_1, b_1), \ldots, (a_m, b_m)\}$ of pairs of vertices from V, and two chosen vertices $s, t \in V$, we must recognize if there exists a directed path from s to t containing at most one vertex from each pair from U. We may suppose that G does not contain edges $(a_1, b_1), \ldots, (a_m, b_m), (b_1, a_1), \ldots, (b_m, a_m)$ (otherwise we can delete them), and the set U does not contain opposite pairs (a_i, b_i) and (b_i, a_i).

Let $V = \{x_1, \ldots, x_n\}$. Let us introduce elements $x_1', \ldots, x_n', z$ and put $A = \{x_1, \ldots, x_n, x_1', \ldots, x_n', z\}$. We shall build a model of sequential choice $\mathbf{r} = r_1 r_2 r_3$ on A by taking $r_1 = \{(x_i', x_i),\ 1 \leqslant i \leqslant n\} \cup U$, $r_2 = G \cup \{(t, s)\}$, $r_3 = \{(x_i, z),\ 1 \leqslant i \leqslant n\}$. It can be easily checked that the model $\mathbf{r}$ is reduced.

Consider an arbitrary reduced model $\mathbf{r}' = r_1' \ldots r_l'$ realizing the choice function $C_{\mathbf{r}}$. Due to Corollary 1, we have $r_1' \cup \cdots \cup r_l' = r_1 \cup r_2 \cup r_3$. For each pair $(x_i, y) \in r_2 \cup r_3$ with $y \in V \cup \{z\}$, there is a pair $(x_i', x_i) \in r_1 \setminus U$ in $\mathbf{r}'$ situated strictly earlier (i.e., in a relation having smaller number) than (x_i, y). It can be easily seen that this fact follows from $C(\{x_i', x_i, y\}) = \{x_i', y\}$. Moreover, each pair $(a_j, b_j) \in U$ is situated in the same relation in $\mathbf{r}'$ as the pair $(a_j', a_j) \in r_1 \setminus U$, because $C(\{a_j', a_j, b_j\}) = \{a_j'\}$ and $(a_j', b_j) \notin r_1 \cup r_2 \cup r_3$. This implies that if there exists a model $\mathbf{r}'$ of depth 2 for the choice function $C_{\mathbf{r}}$, then $\mathbf{r}' = (r_1, r_2 \cup r_3)$.

Let us apply Lemma 8 to the model $\mathbf{r}$. It follows from the form of the relations r_2 and r_3 that $r_2 \cdot r_3 \subseteq r_3$. So, (3) holds. Since the only non-empty set of the form $r_3^{-1}(x)$ is the set $r_3^{-1}(y) = V$, the function $C_{\mathbf{r}}$ can be represented by the model $(r_1, r_2 \cup r_3)$ if and only if there is a cycle in $G \cup \{(t, s)\}$ going through no pair of vertices of U. Since the graph G is acyclic, this cycle must contain the edge (t, s), i.e., there must exist a directed path from s to t in G. Thus, the problem PATH WITH FORBIDDEN PAIRS is reduced to the minimization problem for a model of depth 3, whose NP-hardness is thus proved. The same can be said about the compression problem of depth 3 since the model $(r_1, r_2 \cup r_3)$ is the shortest for $\mathbf{r}$.

Now consider an arbitrary $k > 3$; let $k = 3 + h$. Take the model $\mathbf{r} = r_1 r_2 r_3$ on the set A built above. Introduce the elements $y_1, y_2, \ldots, y_{h+1}$ and put $A' = A \cup \{y_1, \ldots, y_{h+1}\}$. We build the model $\mathbf{r}' = r_1' \ldots r_{h+1}' \hat{r}_1 r_2 r_3$ of depth k on A based on $\mathbf{r}$ as follows: $r_j' = \{(y_j, y_{j+1})\}$, $1 \leqslant j \leqslant h$, $\hat{r}_1 = r_1 \cup \{(y_{h+1}, x_i'),\ 1 \leqslant i \leqslant n\}$. Since $C_{\mathbf{r}'}(\{y_j, y_{j+1}, y_{j+2}\}) = \{y_j, y_{j+2}\}$ and $C_{\mathbf{r}'}(\{y_h, y_{h+1}, x_i'\}) = \{y_h, x_i'\}$, in each model for the choice function $C_{\mathbf{r}'}$ the pair (y_j, y_{j+1}) precedes the pair (y_{j+1}, y_{j+2}), $1 \leqslant j \leqslant h-1$, and the pair (y_h, y_{h+1}) precedes the pairs (y_{h+1}, x_i'), $1 \leqslant i \leqslant n$. This implies that the model $\mathbf{r}'$ can be minimized or compressed if and only if so can $\mathbf{r}$. This fact proves the NP-hardness of these problems for arbitrary $k \geqslant 3$.

(b) Clearly, for $k = 2$ the minimization and compression problems coincide. An effective (polynomial) way to check if the choice function $C_{r_1 r_2}$ can be realized by one relation r ($r = r_1 \cup r_2$) can be based on Lemma 7. Clearly, the conditions (2) can be checked effectively, and the question of existence of a cycle in the graph (or relation) can be solved using the following effective procedure. A vertex of a graph will be called a *source* (a *sink*) if it has no incoming (outgoing) edges. We shall delete successively all sources and sinks of a graph in arbitrary order while it is possible. The initial graph contains a cycle if and only if the graph obtained is not empty. Indeed, if this graph is not empty, then moving arbitrarily from a vertex along edges (while it is possible) and deleting the edges passed, we shall come to a vertex that has no outgoing edges. It means that this vertex has been visited before, i.e., that a directed cycle is passed. □

Remark. (1)Since the model $\mathbf{r} = r_1 r_2 r_3$ built in the (a) part of the proof can be minimized if and only if it is equivalent to the model $(r_1, r_2 \cup r_3)$, or, which is the same, if and only if the model $(r_1, r_2 \cup r_3)$ is embedded to it, the NP-completeness of both the equivalence problem and the embedding problem for such models follows. The indistinguishability problem is polynomial for every k (see Lemma 1).

(2) In [6], the fact that the minimization problem for models of depth 2 is polynomial is proved by different technique using logical representation of choice models.

5. Local procedure of compressing models

The *index* of a local transform $\mathbf{f}' \to \mathbf{f}''$ is the maximum lengths of a sequence of relations $\mathbf{f}'$ and $\mathbf{f}''$. In what follows we shall consider only correct local transforms. Intersecting fragments $\mathbf{f}'$ and $\mathbf{f}''$ of a model $\mathbf{r}$ will be called *chained*.

A *local procedure* consists of steps; at each of them some local transform is applied to a fragment of the model, called *active* at this step, and an active fragment for the next step is chosen among the fragments chained to the currently active fragment. This procedure has a finite number of (*internal*) *states*, one of which is the *terminal* state. Let us describe the tth step $(t = 1, 2, \ldots)$ of the procedure. Let $\mathbf{r}_t$ be the model obtained by the tth step, $\mathbf{f}_t$ be its active fragment, and q_t be the state at the tth step. If q_t is the terminal state, we stop the procedure and consider the model $\mathbf{r}_t$ as its result. Otherwise we point the following functions of the pair $(\mathbf{f}_t, q_t)$:

- a local transform $\mathbf{f}' \to \mathbf{f}''$ $(\mathbf{f}' = \mathbf{f}_t)$ to be executed at the tth step;
- the boundaries of the a active fragment $\mathbf{f}_{t+1}$ chained to $\mathbf{f}''$ in the model $\mathbf{r}_{t+1}$ (obtained from $\mathbf{r}_t$ by applying the local transform $\mathbf{f}' \to \mathbf{f}''$);
- a new state q_{t+1}

and pass to the $(t + 1)$st step. The initial state q_1 and the boundaries of the initial active fragment $\mathbf{f}_1$ are given in advance, and $\mathbf{r}_1$ is the initial model $\mathbf{r}$.

The pair $(\mathbf{r}_t, f_t)$ will be called the *configuration* at tth step and represented by putting the fragment $\mathbf{f}_t$ of the model $\mathbf{r}_t$ in parentheses: for instance, $r_3(r_2 r_3) r_1 r_4$ corresponds to the case of $\mathbf{r}_t = r_3 r_2 r_3 r_1 r_4$, $\mathbf{f}_t = r_2 r_3$.

A *local procedure of index d* is one in which all local transforms have indices at most d. A transformation problem for models will be called

- *d-solvable* if it admits a local procedure of index d;
- *strictly d-solvable* if it is d-solvable but is not $(d - 1)$-solvable;
- *locally unsolvable* if it is not d-solvable for any d.

Lemma 9. *The problems of building the canonical and the reduced models for a given model* $\mathbf{r}$ *are strictly* 2*-solvable.*

Proof. Let $\mathbf{r}=r_1\dots r_k$, $\mathbf{r}^1=r_1^1\dots r_k^1$, and $r=r_1^0\dots r_k^0$ be the initial, the canonical, and the reduced models respectively. Then

$$r_1^1=r_1,\ r_2^1=r_1^1\otimes r_2,\ r_3^1=r_2^1\otimes r_3,\ \dots,\ r_k^1=r_{k-1}\otimes r_k$$

and $\mathbf{r}^1$ can be obtained from $\mathbf{r}$ by applying the transforms of the form $r'r''\to(r',r'\otimes r'')$ while moving the active fragment from left to right. Then, taking into account that

$$r_k^0=r_k^1\setminus r_{k-1}^1,\ r_{k-1}^0=r_{k-1}^1\setminus r_{k-2}^1,\ \dots,\ r_2^0=r_2^1\setminus r_1^1,\ r_1^0=r_1^1,$$

and moving the active fragment from right to left, we can transform the model $\mathbf{r}^1$ to $\mathbf{r}^0$ by transformations of the form $r'r''\to(r',r''\setminus r')$. This means that the problems stated are 2-solvable. The fact that they are not 1-solvable is obvious. This completes the proof. □

Taking Lemma 9 into account, in what follows we deal only with canonical and reduced models. A model $\mathbf{r}$ will be called *redundant* if $C_{\mathbf{r}|_i}=C_{\mathbf{r}|_j}$ for some i and j, $j>i$. In this case, the relations $r_{i+1},\dots,r_j$ can be removed without changing the set $\mathscr{C}_{\mathbf{r}}$. It can be easily shown that the equality $C_{\mathbf{r}|_i}=C_{\mathbf{r}|_j}$ for the reduced model is equivalent to the conditions $r_{i+1}=\cdots=r_j=\emptyset$, and for the canonical model $\mathbf{r}$ it is equivalent to the conditions $r_i=r_{i+1}=\cdots=r_j$. It is easy to see that the redundant relations $r_{i+1},\dots,r_j$ can be removed from the canonical or reduced model by a local procedure of index 2. It what follows, we assume that the models are not redundant.

If $\mathbf{r}$, $\mathbf{r}'$ are canonical models and $\mathbf{r}'$ is embedded to $\mathbf{r}$, then (assuming they are not redundant) according to Lemma 4 we have

$$\mathbf{r}'=r_{i_1}r_{i_2}\dots r_{i_u},\quad 1\leqslant i_1<i_2<\cdots<i_u\leqslant k, \tag{5}$$

i.e., $\mathbf{r}'$ can be obtained from $\mathbf{r}$ by removing fragments $\mathbf{r}|_{i_j}^{i_{j+1}-1}$, $i_{j+1}>i_j+1$. A fragment $\mathbf{r}|_i^j$, $j<k$, of a canonical non-redundant model $\mathbf{r}$ will be called *removable* if the model $\mathbf{r}'=(\mathbf{r}|_{i-1},\mathbf{r}|_{j+1}^k)$ is embedded to $\mathbf{r}$ (it follows from Lemma 3 and $j<k$ that the models $\mathbf{r}$ and $\mathbf{r}'$ are equivalent). We can speak also about removable relations considering them as fragments of length 1.

Lemma 10. *If a fragment* $\mathbf{r}|_i^j$ *of a canonical model* $\mathbf{r}$ *is removable, then the relation* r_j *is removable from* $\mathbf{r}$.

Proof. Put $\mathbf{r}_1=\mathbf{r}|_{i-1}$ and $\mathbf{r}_2=\mathbf{r}|_i^{j-1}$. It is sufficient to prove the equivalence of models $\mathbf{r}_1r_{j+1}$ and $\mathbf{r}_1\mathbf{r}_2r_{j+1}$.

It follows from Lemma 3 and the equalities $r_j\otimes r_{j+1}=r_{j+1}$, $\otimes(\mathbf{r}_2r_{j+1})=r_{j+1}$ that $C_{r_{j+1}}$ is the lower approximation of the functions $C_{r_jr_{j+1}}$ and $C_{\mathbf{r}_2r_{j+1}}$. That is why $C_{r_jr_{j+1}}\geqslant C_{r_{j+1}}$ and $C_{\mathbf{r}_2r_{j+1}}\geqslant C_{r_{j+1}}$. Thus,

$$C_{\mathbf{r}_2r_jr_{j+1}}=C_{\mathbf{r}_2}\circ C_{r_jr_{j+1}}\geqslant C_{\mathbf{r}_2}\circ C_{r_{j+1}}=C_{\mathbf{r}_2r_{j+1}}\geqslant C_{r_{j+1}}.$$

This implies the relations

$$C_{\mathbf{r}_1\mathbf{r}_2 r_j r_{j+1}} \geqslant C_{\mathbf{r}_1\mathbf{r}_2 r_{j+1}} \geqslant C_{\mathbf{r}_1 r_{j+1}}.$$

Since the fragment $\mathbf{r}_2 r_j = \mathbf{r}|_i^j$ is removable, the models $\mathbf{r}_1\mathbf{r}_2 r_j r_{j+1}$ and $\mathbf{r}_1 r_{j+1}$ are equivalent, and the inequalities can be replaced by equalities. This implies $C_{\mathbf{r}_1\mathbf{r}_2 r_{j+1}} = C_{\mathbf{r}_1 r_{j+1}}$, which means the equivalence of $\mathbf{r}_1 r_{j+1}$ and $\mathbf{r}_1\mathbf{r}_2 r_{j+1}$. $\square$

Remark. Lemma 10 assures that the last relation r_j of the fragment $\mathbf{r}|_i^j$ is removable from $\mathbf{r}$. The first relation r_i can be not removable. Consider the canonical model $\mathbf{r} = r_1 r_2 r_3$ on the set $A = \{x, y, z\}$ defined by $r_1 = \{(y,x)\}$, $r_2 = \{(y,x),(x,z)\}$, $r_3 = \{(y,x),(x,z),(y,z),(z,y)\}$. It can be checked that $C_{\mathbf{r}} = C_{r_3}$, and thus the fragment $r_1 r_2$ is removable from $\mathbf{r}$ (according to the lemma, the same can be said concerning r_2). But r_1 is not removable since $C_{\mathbf{r}}(A) = \emptyset \neq C_{r_2 r_3}(A) = \{y\}$.

Lemma 11. *A relation r_i, $1 \leqslant i \leqslant k-1$, of a canonical model $\mathbf{r} = r_1 \dots r_k$ is removable if and only if r_i is removable from the model $\mathbf{r}' = r_{i-1} r_i r_{i+1}$ of depth* 3, *where $r_0 = \emptyset$.*

Proof. Let r_i be removable from $\mathbf{r}$. Consider an $X \subseteq A$ and put $Y = C_{r_{i-1}}(X)$. Since $C_{r_{i-1}}(Y) = Y$, and $C_{r_{i-1}}$ is a lower approximation of the function $C_{\mathbf{r}|_{i-1}}$, we have $Y \supseteq C_{\mathbf{r}|_{i-1}}(Y) \supseteq Y$, i.e., $C_{\mathbf{r}|_{i-1}}(Y) = Y$. Taking into account that r_i is removable from $\mathbf{r}$, we obtain

$$\begin{aligned} C_{\mathbf{r}'}(X) &= C_{r_{i+1}}(C_{r_i}(C_{r_{i-1}}(X)) = C_{r_{i+1}}(C_{r_i}(Y)) = C_{r_{i+1}}(C_{r_i}(C_{\mathbf{r}|_{i-1}}(Y))) \\ &= C_{\mathbf{r}|_{i+1}}(Y) = C_{r_{i+1}}(C_{\mathbf{r}|_{i-1}}(Y)) = C_{r_{i+1}}(Y) = C_{r_{i+1}}(C_{r_{i-1}}(X)). \end{aligned}$$

Since X is arbitrary, this means that r_i is removable from $\mathbf{r}'$.

Conversely, if r_i is removable from $\mathbf{r}'$, then

$$C_{\mathbf{r}|_{i+1}} = C_{\mathbf{r}|_{i-2}} \circ C_{\mathbf{r}'} = C_{\mathbf{r}|_{i-2}} \circ C_{r_{i-1} r_{i+1}} = C_{\mathbf{r}_{i-1} r_{i+1}}.$$

This implies that r_i is removable from $\mathbf{r}$. $\square$

For a canonical model $\mathbf{r}$, let us denote by $\mathscr{R} = \mathscr{R}(\mathbf{r})$ the set of models embedded to $\mathbf{r}$ and equivalent to it. The embedded models look as (5); we shall denote them by $\mathbf{r}[I]$, where $I = i_1 i_2 \dots i_u$. Since $\mathbf{r}$ and $\mathbf{r}[I]$ are equivalent and due to Corollary 1, we have $i_u = k$. For a number h, $1 \leqslant h \leqslant k$, and a model $\mathbf{r}[I]$, let us define the *left cut* $\mathbf{r}[I,h] = r_{i_1} \dots r_{i_a}$ and the *right cut* $\mathbf{r}[h,I] = r_{i_{a+1}} \dots r_{i_u}$, where $a = \mu(h,I)$, $\mu(h,I) = \max\{s \mid i_s \leqslant h,\ 1 \leqslant s \leqslant u\}$. In the case of $i_1 > h$ ($i_u \leqslant h$), the left (the right) cut is assumed to be empty. Given models $\mathbf{r}[I]$, $\mathbf{r}[J]$ and a number h, we define the model $\mathbf{r}[I,h,J] = \mathbf{r}[I,h]\mathbf{r}[h,J]$ as obtained by adding the right cut of $\mathbf{r}[J]$ to the left cut of $\mathbf{r}[I]$.

Lemma 12. *If $\mathbf{r}[I]$, $\mathbf{r}[J] \in \mathscr{R}$ and $i_{\mu(h,I)} \geqslant j_{\mu(h,J)}$, then $\mathbf{r}[I,h,J] \in \mathscr{R}$.*

Proof. Let

$$I = i_1 i_2 \dots i_u, \quad 1 \leqslant i_1 < i_2 < \cdots < i_u = k, \tag{6}$$

$$J = j_1 j_2 \dots j_v, \quad 1 \leqslant j_1 < j_2 < \cdots < j_v = k, \tag{7}$$

$a = \mu(h, I)$, $b = \mu(h, J)$. Then $j_b \leqslant i_a \leqslant h < j_{b+1}$
and

$$\mathbf{r}[I, h, J] = r_{i_1} \dots r_{i_a} r_{j_{b+1}} \dots r_{j_v}.$$

Put $K = \{1, \dots, k\}$ and consider the model

$$\mathbf{r}' = \mathbf{r}[I, i_a, K] = r_{i_1} \dots r_{i_a} r_{i_a+1} \dots r_k = r_{i_1} \dots r_{i_a} \dots r_{j_{b+1}} \dots r_k.$$

For $s \leqslant a$, we have $\mathbf{r}'|_s = \mathbf{r}[I]|_s$. Since $\mathbf{r}[I] \in \mathscr{R}$, the functions $C_{\mathbf{r}'|_s}$ are contained in $\mathscr{C}_{\mathbf{r}}$. Here (as in the proof of Lemma 4) $C_{\mathbf{r}'|_a} = C_{\mathbf{r}|_{i_a}}$. Then, we have

$$C_{\mathbf{r}'|_{a+1}} = C_{r_{i_a+1}}(C_{\mathbf{r}'|_a}) = C_{r_{i_a+1}}(C_{\mathbf{r}|_{i_a}}) = C_{\mathbf{r}|_{i_a+1}}$$

and so on, until we obtain $C_{\mathbf{r}'} = C_{\mathbf{r}'|_{k-i_a+a}} = C_{\mathbf{r}|_k} = C_{\mathbf{r}}$. Thus, the model $\mathbf{r}'$ is embedded to $\mathbf{r}$ and is equivalent to it, i.e., $\mathbf{r}' \in \mathscr{R}$.

Now let us consider the model

$$\mathbf{r}'' = \mathbf{r}[J, j_b, K] = r_{j_1} \dots r_{j_b} r_{j_b+1} \dots r_k.$$

It follows from $\mathbf{r}[J] \in \mathscr{R}$ and (7) that the fragment $r_{j_b+1} \dots r_{j_{b+1}-1}$ is removable from $\mathbf{r}''$. Due to Lemma 10, it follows that the relation $r_{i_{b+1}-1}$ is removable from $\mathbf{r}''$. Due to Lemma 11, this fact is determined by the triple $(r_{j_{b+1}-2}, r_{j_{b+1}-1}, r_{j_{b+1}})$. Since this triple appears in $\mathbf{r}'$, the relation $r_{j_{b+1}-1}$ is removable from $\mathbf{r}'$. Applying the same arguments to the models obtained from $\mathbf{r}'$ and $\mathbf{r}''$ by removing $r_{j_{b+1}-1}$, we can prove that the relation $r_{j_{b+1}-2}$ is also removable from them. Since $j_b \leqslant i_a$, this procedure can be continued until removing r_{i_a+1}. As the result, we obtain the model $r_{i_1} \dots r_{i_a} r_{j_{b+1}} r_{j_{b+1}+1} \dots r_k \in \mathscr{R}$. Analogously, after removing all the non-empty fragments $r_{j_s+1} \dots r_{j_{s+1}-1}$, $b+1 \leqslant s \leqslant v-1$, we obtain $\mathbf{r}[I, h, J]$. So, $\mathbf{r}[I, h, J] \in \mathscr{R}$. $\square$

Let us introduce the set $\mathscr{I} = \mathscr{I}(\mathbf{r}) = \{I \mid \mathbf{r}[I] \in \mathscr{R}\}$ and order $\mathscr{I}$ lexicographically by putting for the sets (6) and (7)

$$I \succ J \Leftrightarrow i_1 > j_1 \vee (i_1 = j_1,\, i_2 > j_2) \vee (i_1 = j_1,\, i_2 = j_2,\, i_3 > j_3) \vee \dots$$

Clearly, the condition of completeness $I \neq J \Rightarrow I \succ J \vee J \succ I$ holds, and thus this is a strict linear order. In particular, there exists the maximal set $I_0 : (\forall I \in \mathscr{I}) I_0 \succ I$.

Lemma 13. *The embedded model* $\mathbf{r}[I_0]$ *corresponding to the maximal* I_0 *is the shortest for the model* $\mathbf{r}$.

Proof. It is sufficient to prove that for each model $\mathbf{r}[J] \in \mathscr{R}$, $J \neq I_0$, there exists a model $\mathbf{r}[J'] \in \mathscr{R}$, $J' \succ J$, whose depth is not greater than that of $\mathbf{r}[J]$.

Let I be a set from $\mathscr{I}$ such that $I \succ J$, and let I and J be of the form (6) and (7). We find a and b from the conditions

$$a = \min\{s \mid i_s > j_s,\ 1 \leqslant s \leqslant u\}, \quad b = \min\{s \mid j_s > i_a,\ a < s \leqslant v\}$$

and create the set $J' = (i_1, \dots, i_a, j_b, \dots, j_v)$. It follows from $i_s = j_s$, $1 \leqslant s \leqslant a-1$, and $i_a > j_a$ that $J' \succ J$. The length $v-(b-a)+1$ of J' is not greater than the length v of J. The model $\mathbf{r}[J']$ coincides with $\mathbf{r}[I, i_a, J]$. Thus, taking into account that $i_a = \mu(i_a, I) \geqslant \mu(i_a, J) = j_{b-1}$ and Lemma 12, we conclude that $\mathbf{r}[J'] \in \mathscr{R}$. $\square$

The main result of this section is

Theorem 2. *The compression problem for models of sequential choice is strictly* 3-*solvable*.

Proof. We solve the compression problem by building the model $\mathbf{r}[I_0]$ from Lemma 13. Let $\Delta(i, \mathbf{r})$, $1 \leqslant i \leqslant k$ denote the removable fragment $\mathbf{r}|_i^j$ of a (canonical) model $\mathbf{r}$ having the left boundary i and the maximal possible right bound $j = j(i, \mathbf{r})$. (If $\Delta(i, \mathbf{r}) = \emptyset$, we put $j(i, \mathbf{r}) = i-1$.) We shall build the model $\mathbf{r}[I_0]$ by removing successively fragments $\Delta_1, \Delta_2, \dots, \Delta_s, \dots$ from $\mathbf{r}$, where $\Delta_1 = \Delta(1, \mathbf{r})$, $\Delta_s = \Delta(j_{s-1}+1, \mathbf{r})$, and $j_{s-1}-1$ is the right boundary of the fragment Δ_{s-1}. According to Lemma 10, removing the fragment Δ_s can be done by successively excluding the relations r_{j_s-1}, r_{j_s-2}, $\dots$, $r_{j_{s-1}+1}$. The right boundary $j_s - 1$ of the fragment Δ_s can be found by looking through the values of $j = k-1, k-2, \dots$ and building for each j the maximal possible sequence of removable relations $r_j, r_{j-1}, \dots$. The value of $j_s - 1$ needed coincides with the first j met such that this sequence includes $r_{j_{s-1}+1}$.

The local procedure of index 3 realizing this plan is divided into *stages* s, $s = 1, 2, \dots$ consisting of *visits* (s, j), $j = k-1, k-2, \dots$. The goal of the stage s is to remove the fragment Δ_s. At the visit (s, j) we find the maximal removable fragment ending with r_j.

Let us pass to the description of the procedure. We shall not indicate states; they correspond to types of behaviour in different situations whose number is clearly finite. We can check directly that the transforms used in the local procedure are correct.

Procedure: The initial configuration is $(r_1)r_2 \dots r_k$. At the step 1, we apply the local transform $r_1 \to r_0 r_0 r_1$ where r_0 is the empty relation and start the stage 1 at the configuration $(r_0 r_0 r_1) r_2 \dots r_k$.

Stage s: At the beginning of stage s, the configuration is of the form

$$r_{j_1} \dots r_{j_{s-2}} (r_{j_{s-1}} r_{j_{s-1}} r_{j_{s-1}+1}) r_{j_{s-1}+2} \dots r_k.$$

The changes at the stage s are committed in the area $r_{j_{s-1}} r_{j_{s-1}} r_{j_{s-1}+1} \dots r_k$. First, the active fragment moves sequentially to the end without changes in the current model. After the fragment $r_{k-2} r_{k-1} r_k$ becomes active, we start the visit $(s, k-1)$.

Visit (s, j): The initial configuration of the visit is

$$r_{j_1} \dots r_{j_{s-2}} r_{j_{s-1}} r_{j_{s-1}} r_{j_{s-1}+1} \dots r_{j-2} (r_{j-1} r_j r_{j+1}) r_{j+2} \dots r_k.$$

(1) If $j = j_{s-1} + 1$, then the configuration is of the form

$$r_{j_1} \dots r_{j_{s-2}} r_{j_{s-1}} (r_{j_{s-1}} r_j r_{j+1}) r_{j+2} \dots r_k.$$

Here

(a) if r_j is not removable from $r_{j_{s-1}}r_jr_{j+1}$, then the fragment $r_{j_{s-1}}r_{j_{s-1}}r_j$ becomes active, the local transform $r_{j_{s-1}}r_{j_{s-1}}r_j \to r_{j_{s-1}}r_jr_j$ is carried out, and after passing to the fragment $r_jr_{j+1}r_{j+2}$ the stage $s+1$ begins, so $j_s = j$;

(b) if r_j is removable from $r_{j_{s-1}}r_jr_{j+1}$, then the transforms $r_{j_{s-1}}r_jr_{j+1} \to r_{j_{s-1}}r_{j+1}$, $r_{j_{s-1}}r_{j_{s-1}}r_{j+1} \to r_{j_{s-1}}r_{j+1}r_{j+1}$ are carried out, and after passing to the fragment $r_{j+1}r_{j+2}r_{j+3}$ the stage $s+1$ begins, so $j_s = j+1$.

(2) Let $j > j_{s-1}+1$. If r_j is removable from $r_{j-1}r_jr_{j+1}$, then the local transform $r_{j-1}r_jr_{j+1} \to r_{j-1}r_{j+1}r_j$ is carried out, and the triple $r_{j-2}r_{j-1}r_{j+1}$ becomes active. If r_{j-1} is removable from it, then the transform $r_{j-2}r_{j-1}r_{j+1} \to r_{j-2}r_{j+1}r_{j-1}$ is carried out, the triple $r_{j-3}r_{j-2}r_{j+1}$ becomes active, and so on. Two cases are possible:

(a) The chain of removable relations does not reach $r_{j_{s-1}+1}$. In this case, the configuration takes the form

$$r_{j_1}\dots r_{j_{s-1}+1}\dots(r_ur_{u+1}r_{j+1})r_{u+2}\dots r_jr_{j+2}\dots r_k,$$

where r_{u+1} is not removable from the triple. At the next step, the fragment $r_{u+1}r_{j+1}r_{u+2}$ becomes active, and then the relation r_{j+1} is moved to its initial position by the chain of transforms $r_{u+1}r_{j+1}r_{u+2} \to r_{u+1}r_{u+2}r_{j+1}, \dots, r_{j-1}r_{j+1}r_j \to r_{j-1}r_jr_{j+1}$ (this is indicated by the relation $r_{j+1} \subset r_{j+2}$ to be checked at the next step). Then the fragment $r_{j-2}r_{j-1}r_j$ becomes active and the visit $(s, j-1)$ begins.

(b) The chain of removable relations includes $r_{j_{s-1}+1}$. Then the configuration

$$r_{j_1}\dots(r_{j_{s-1}}r_{j_{s-1}}r_{j+1})r_{j_{s-1}+1}\dots r_k$$

is reached. After the local transform $r_{j_{s-1}}r_{j_{s-1}}r_{j+1} \to r_{j_{s-1}}r_{j+1}$ the movement to the right with deleting removable relations $r_{j_{s-1}+1},\dots,r_j$ is started. It is realized by the transforms $r_{j_{s-1}}r_{j+1}r_{j_{s-1}+1} \to r_{j_{s-1}}r_{j+1}, \dots, r_{j_{s-1}}r_{j+1}r_j \to r_{j_{s-1}}r_{j+1}$. At the next right shift, the fragment $r_{j_{s-1}}r_{j+1}r_{j+2}$ becomes active (the inclusion $r_{j+1} \subset r_{j+2}$ indicates that the relation r_{j+1} took its place). The local transform $r_{j+1}r_{j+2} \to r_{j+1}r_{j+1}r_{j+2}$ (taking into account that $r_{j+1} = r_{j_s}$) leads to the configuration $r_{j_1}\dots r_{j_{s-1}}(r_{j_s}r_{j_s}r_{j_s+1})r_{j_{s+2}}\dots r_k$ where stage $s+1$ starts.

If after finishing stage s it turns out that $j_s+1=k$, then the local transform $r_{j_s}r_{j_s}r_k \to r_{j_s}r_k$ is applied and the procedure stops.

We have proved that the compression problem is 3-solvable. To prove that it is not solvable by procedures of index 2, consider the example from the remark to Lemma 10. It corresponds to the reduced model $\mathbf{r} = r_1r_2r_3$, $r_1 = \{(y,x)\}$, $r_2 = \{(x,z)\}$, $r_3 = \{(y,z),(z,y)\}$, $A = \{x,y,z\}$. Let a local procedure of index 2 be applied to $\mathbf{r}$, let $\mathbf{r}_t$ be the current model at the step t, and $\hat{\mathbf{r}}_t$ be the corresponding reduced model. By a_t (b_t) denote the number of the relation in the model $\hat{\mathbf{r}}_t$ containing the pair (x,z) (the pairs (y,z), (z,y)). Let us show that $a_t < b_t$ for all t.

Assume that this does not hold and that t is the first value with $a_t \geqslant b_t$. For the model $\mathbf{r}_1 = \mathbf{r}$ we have $a_1 < b_1$, so $t \geqslant 2$. Clearly, $a_{t-1} < b_{t-1}$. Let the model $\mathbf{r}_t$ be obtained from $\mathbf{r}_{t-1}$ by the transform $r_\alpha r_\beta \to r_\gamma r_\delta$. We shall prove that it is not correct, i.e., that the models $r_\alpha r_\beta$ and $r_\gamma r_\delta$ are not equivalent. Instead of them, we can consider the reduced models $\hat{r}_\alpha\hat{r}_\beta$ and $\hat{r}_\gamma\hat{r}_\delta$. Clearly, $(x,z)\in\hat{r}_\alpha,\hat{r}_\delta$ and $(y,z),(z,y)\in\hat{r}_\beta,\hat{r}_\gamma$. Besides, some of the relations $\hat{r}_\alpha,\hat{r}_\beta,\hat{r}_\gamma,\hat{r}_\delta$ may contain the pair (y,x). The fact that the models $\hat{r}_\alpha\hat{r}_\beta$ and $\hat{r}_\gamma\hat{r}_\delta$ are not equivalent follows from the correlations $y\in C_{\hat{r}_\alpha\hat{r}_\beta}(\{x,y,z\})$ and $y\notin C_{\hat{r}_\gamma\hat{r}_\delta}(\{x,y,z\})$, which can be checked directly. Analogously, we consider the case when $\mathbf{r}_t$ is obtained from $\mathbf{r}_{t-1}$ by means of the substitution $r_\alpha r_\beta\to r_\gamma$ (this is the case of $a_t=b_t$).

It can be easily checked that the initial model $\mathbf{r}$ can be realized by the relation $r=\{(y,x),(x,z),(y,z),(z,y)\}$, which is a unique solution of the compression problem. But it cannot be obtained by a procedure of index 2 because it violates the conditions of $a_t<b_t$. □

In applications, acyclic relations are those mainly used. The next corollary follows from the construction in Theorem 2:

Corollary 3. *The compression problem for models using only acyclic relations is polynomial.*

It can be easily seen that the local procedure from Theorem 2 is realized in polynomial number of steps. The time complexity of each step is determined by the time of checking if the relations from the triple are removable. For acyclic relations, this time is polynomial since the reduced model also consists of acyclic relations, and we just need to check the condition (3).

In the general case, the compression problem is NP-hard (Theorem 1).

6. The minimization problem is locally unsolvable

It follows from Theorem 1 that there is no much difference between the time complexity of the minimization problem and that of the compression problem: they are polynomially reducible to each other. In this section, we prove that from the informational point of view, they behave substantially different. Let us say that a model is *d-irreducible* if its depth cannot be decreased by procedures of index d.

Theorem 3. *For each d and k, $3\leqslant d<k$, there exists a model of depth k which is d-irreducible and has an equivalent minimal model of depth d which can be obtained by a procedure of index $d+1$.*

Proof. Put $s=k-d$ and

$$A=\{\alpha_1,\dots,\alpha_s,\beta_1,\dots,\beta_s,\gamma_1,\dots,\gamma_s,\sigma_1,\dots,\sigma_{d-1},\tau_1,\dots,\tau_{d-1}\}$$

and consider the model $\mathbf{r}=r_1r_2\dots r_{s+d}$ on A depicted in Fig. 1 (two-sided arrows denote pairs of opposite edges). Here

- $r_1=\{(\alpha_1,\beta_1),(\beta_1,\sigma_j),\ 1\leqslant j\leqslant d-1\}$;
- $r_i=\{(\alpha_i,\beta_i),(\beta_i,\beta_{i-1}),(\beta_{i-1},\gamma_{i-1}),(\beta_i,\sigma_j),\ 1\leqslant j\leqslant d-1\},\ 2\leqslant i\leqslant s$;
- $r_{s+1}=\{(\beta_s,\gamma_s)\}$;

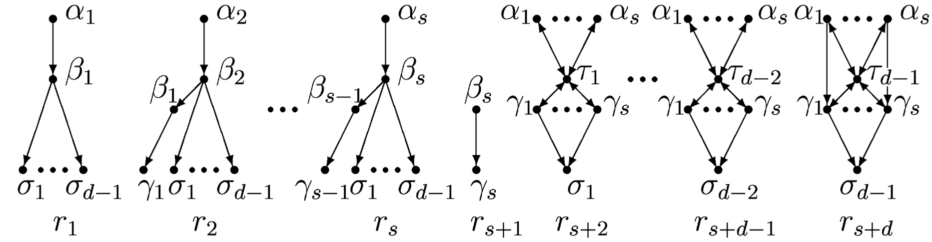

Fig. 1.

- $r_{s+j+1} = \{(\alpha_i, \tau_j), (\tau_j, \alpha_i), (\gamma_i, \tau_j), (\tau_j, \gamma_i), (\gamma_i, \sigma_j),\ 1 \leqslant i \leqslant s\},\ 1 \leqslant j \leqslant d-2$;
- $r_{s+d} = \{(\alpha_i, \tau_{d-1}), (\tau_{d-1}, \alpha_i), (\gamma_i, \tau_{d-1}), (\tau_{d-1}, \gamma_i), (\gamma_i, \sigma_{d-1}), (\alpha_i, \gamma_i),\ 1 \leqslant i \leqslant s\}$.

This model is reduced. According to Corollary 1, each reduced model $\mathbf{r}'$ realizing the choice function $C_{\mathbf{r}'}$ satisfies $\cup\mathbf{r}' = \cup\mathbf{r}$. For $(x, y) \in \cup\,\mathbf{r}$ and $\mathbf{r}'$, denote by $\#(x, y)_{\mathbf{r}'}$ the number of the relation in $\mathbf{r}'$ containing the pair (x, y); sometimes, if it is clear which model $\mathbf{r}'$ is considered, we shall write just $\#(x, y)$.

Let a local procedure of index d be applied to the model $\mathbf{r}$, let $\mathbf{r}_t$ be the current model at the step t, and $\hat{\mathbf{r}}_t$ be the corresponding reduced model. Since all transforms are correct, the models $\mathbf{r}_t$ and $\hat{\mathbf{r}}_t$ are equivalent to the model $\mathbf{r}$ and therefore $\cup\hat{\mathbf{r}}_t = \cup\mathbf{r}$.

We shall show by induction on t that for $\hat{\mathbf{r}}_t$, the relations hold

$$\#(\alpha_i, \beta_i) < \#(\beta_i, \gamma_i), \quad 1 \leqslant i \leqslant s. \tag{8}$$

The model $\hat{\mathbf{r}}_1$ coincides with $\mathbf{r}$, and for it, the relations hold. Suppose that they hold for $\hat{\mathbf{r}}_{t-1}$. Then for each i we have the following relations in $\hat{\mathbf{r}}_{t-1}$:

$$\begin{aligned}
\#(\alpha_i, \beta_i) &< \#(\beta_i, \gamma_i) < [\gamma_i \in C_{\mathbf{r}}(\{\beta_i, \gamma_i, \sigma_1\})] < \#(\gamma_i, \sigma_1) \\
&\leqslant [\tau_1 \in C_{\mathbf{r}}(\{\gamma_i, \sigma_1, \tau_1\})] \leqslant \#(\tau_1, \gamma_i) < [\sigma_2 \in C_{\mathbf{r}}(\{\gamma_i, \tau_1, \sigma_2\})] < \#(\gamma_i, \sigma_2) \\
&\leqslant [\text{analogously}] \leqslant \#(\tau_2, \gamma_i) < \#(\gamma_i, \sigma_2) \leqslant \cdots < \#(\gamma_i, \sigma_{d-1}) \\
&\leqslant [\sigma_{d-1} \notin C_{\mathbf{r}}(\{\alpha_i, \gamma_i, \sigma_{d-1}\})] \leqslant \#(\alpha_i, \gamma_i).
\end{aligned} \tag{9}$$

In the square brackets, there are comments to the inequalities. For example, the inclusion $\gamma_i \in C_{\mathbf{r}}(\{\beta_i, \gamma_i, \sigma_1\})$ which can be checked directly by Fig. 1, assures the inequality $\#(\beta_i, \gamma_i) < \#(\gamma_i, \sigma_1)$, for in the case of $\#(\beta_i, \gamma_i) \geqslant \#(\gamma_i, \sigma_1)$ we have $\gamma_i \notin C_{\hat{\mathbf{r}}_{t-1}}(\{\beta_i, \gamma_i, \sigma_1\})$, which contradicts the fact that $\hat{\mathbf{r}}_{t-1}$ and $\mathbf{r}$ are equivalent.

Let the model $\mathbf{r}_t$ be obtained from $\mathbf{r}_{t-1}$ by applying a local transform $\mathbf{f}' \to \mathbf{f}''$. Suppose that (8) does not hold for $\hat{\mathbf{r}}_t$, i.e., for some i we have $\#(\alpha_i, \beta_i)_{\hat{\mathbf{r}}_t} \geqslant \#(\beta_i, \gamma_i)_{\hat{\mathbf{r}}_t}$. Taking into account (8) for $\hat{\mathbf{r}}_{t-1}$, we conclude that the first occurrences of the pairs (α_i, β_i) and (β_i, γ_i) to $\mathbf{r}_{t-1}$ are situated in $\mathbf{f}'$.

Due to (9), the pair (α_i, γ_i) is separated from (α_i, β_i) in $\mathbf{r}_{t-1}$ by at least $d-1$ intermediate relations and thus cannot get into the fragment of $\mathbf{f}'$ of length d. This and (8) imply $C_{\mathbf{f}'}(\{\alpha_i, \beta_i, \gamma_i\}) = \{\alpha_i, \gamma_i\}$. At the same time, $\#(\alpha_i, \beta_i)_{\hat{\mathbf{r}}_t} \geqslant \#(\beta_i, \gamma_i)_{\hat{\mathbf{r}}_t}$ implies $\gamma_i \notin C_{\mathbf{f}''}(\{\alpha_i, \beta_i, \gamma_i\})$. This means that the models $\mathbf{f}'$ and $\mathbf{f}''$ are not equivalent and thus

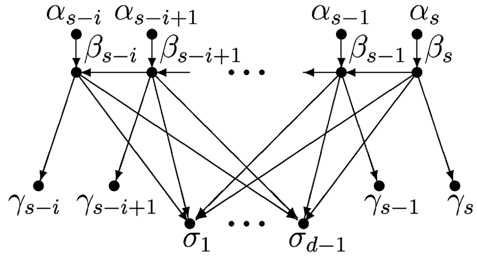

Fig. 2.

the transform $\mathbf{f}' \to \mathbf{f}''$ is not correct. This contradiction completes the induction step and proves (8) for $\hat{\mathbf{r}}_t$ for each t.

It follows from (8) that

$$\#(\alpha_i,\beta_i) < \#(\beta_i,\gamma_i) \leqslant [\gamma_i \notin C_{\mathbf{r}}(\{\beta_i,\beta_{i+1},\gamma_i\})] \leqslant \#(\beta_{i+1},\beta_i)$$
$$\leqslant [\beta_i \notin C_{\mathbf{r}}(\{\alpha_{i+1},\beta_i,\beta_{i+1}\})] \leqslant \ (\alpha_{i+1},\beta_{i+1}).$$

Combined with (9) applied to $\hat{\mathbf{r}}_t$, this yields

$$\#(\alpha_1,\beta_1) < \cdots < \#(\alpha_s,\beta_s) < \#(\beta_s,\gamma_s) < \#(\gamma_s,\sigma_1) < \cdots < \#(\gamma_s,\sigma_{d-1}).$$

Thus, at each step t the depth of $\mathbf{r}_t$, which is equal to the depth of $\hat{\mathbf{r}}_t$, is at least $s+d=k$. This means that $\mathbf{r}$ is d-irreducible.

For each i, $0 \leqslant i \leqslant s$, let us introduce the relation

$$r^{(i)} = \bigcup_{0\leqslant u\leqslant i} r_{s+1-u} \quad (r^{(0)} = r_{s+1})$$

(depicted in Fig. 2) and consider the model $\mathbf{r}^{(i)} = r^{(i)} r_{s+2} \ldots r_{s+d}$. We shall show by induction on i ($1 \leqslant i \leqslant s$) that the models $\mathbf{r}^{(i)}$ and $r_{s+1-i}\mathbf{r}^{(i-1)}$ are equivalent, i.e., that the corresponding choice functions denoted by C_i and C_i' are equal.

The base of induction ($i=1$) is analogous to the induction step, so we start with the passage from i to $i+1$, i.e., with the proof that the models $\mathbf{r}^{i+1}$ and $r_{s-i}\mathbf{r}^{(i)}$ are equivalent. Consider an arbitrary $X \subseteq A$.

Case 1: ($\beta_{s-i} \notin X$). In this case, the relation $r^{(i+1)}|_X$ (the contraction of $r^{(i+1)}$ to X) is split to disconnected parts $r^{(i)}|_X$ and $r_{s-i}|_X$. Thus, $C_{r^{(i+1)}}(X) = C_{r_{s-i}r^{(i)}}(X)$, which implies $C_{i+1}(X) = C'_{i+1}(X)$.

Case 2: ($\beta_{s-i} \in X$). By $T = \{\tau_{j_1},\ldots,\tau_{j_s}\}$, $j_1 < \cdots < j_s$, denote the set of all elements τ_j occurring in X. Put $X' = X \setminus \{\sigma_1,\ldots,\sigma_{d-1}\}$. Since pairs of the form (σ_j, x) do not occur in the models, and the relations $r^{(i+1)}$ and r_{s-i} contain pairs (β_{s-i},σ_j) for all j, it follows that $C_{i+1}(X) = C_{i+1}(X')$, $C'_{i+1}(X) = C'_{i+1}(X')$, and instead of X we can consider X'. If $\tau_j \notin T$ then for $j \leqslant d-2$ the relation $r_{s+j+1}|_X$ is empty (and can be omitted), and for $j = d-1$ it consists only of the pairs of the form (α_i,γ_i). Denote by Y and Y' the sets of elements chosen from X' by the relation $r^{(i+1)} = r^{(i)} \cup r_{s-i}$

and the model $r^{(i)}r_{s-i}$, respectively. We can see from Figs. 1 and 2 that Y' differs from Y only in the case of $\alpha_{s-i}, \beta_{s-i}, \gamma_{s-i} \in X'$. Here $Y' \setminus Y = \{\gamma_{s-i}\}$ and $\alpha_{s-i} \in Y, Y'$. It follows from what has been said above on the relations $r_{s+j+1}|_{X'}$ that if $T \neq \emptyset$, then $C_{i+1}(X) = C_{i+1}(X') = T \setminus \{\tau_{j_1}\}$. But if $T = \emptyset$, then the sets $C_{i+1}(X)$ and $C_{i+1}(X')$ are chosen from Y and Y' with the use of the relation $r_{s+d}|_{X'}$ containing in particular the pair $(\alpha_{s-i}, \gamma_{s-i})$. Furthermore, the element γ_{s-i} will be removed, which will imply $C_{i+1}(X') = C'_{i+1}(X')$ and thus $C_{i+1}(X) = C'_{i+1}(X)$.

The induction is completed. Consider the following local procedure of index $d+1$. To the model $\mathbf{r}$, which can be written as $r_1 \dots r_s \mathbf{r}^{(0)}$, we apply the local transform $r_s\mathbf{r}^{(0)} \to \mathbf{r}^{(1)}$, to the model $r_1 \dots r_{s-1}\mathbf{r}^{(1)}$ obtained we apply the local transform $r_{s-1}\mathbf{r}^{(1)} \to \mathbf{r}^{(2)}$, and so on. After applying the last local transform $r_1\mathbf{r}^{(s-1)} \to \mathbf{r}^{(s)}$, we shall obtain the model $\mathbf{r}^{(s)} = r^{(s)}r_{s+2} \dots r_{s+d} = (r_1 \cup \dots \cup r_s \cup r_{s+1})r_{s+2} \dots r_{s+d}$ of depth d. We have shown before that these transforms are correct, and thus that the model $\mathbf{r}^{(s)}$ obtained is equivalent to the initial model $\mathbf{r}$. According to (9), for each reduced model equivalent to $\mathbf{r}$ we have

$$\#(\beta_s, \gamma_s) < \#(\gamma_s, \sigma_1) < \#(\gamma_s, \sigma_2) \dots < \#(\gamma_s, \sigma_{d-1}).$$

This means that the depth of any model equivalent to $\mathbf{r}$ is at least d, and thus the model $\mathbf{r}^{(s)}$ is minimal. □

Remark. (1) For each $k \geqslant 4$ and $d = k - 1$, the construction from Theorem 3 gives the model of depth k which cannot be simplified by procedures dealing with its fragments distinct from the whole model. The minimization problem can be solved only by considering the model as a whole.

(2) Theorem 3 shows that for each d=const$\geqslant 3$, the procedures of index $d + 1$ are much stronger then those of index d: there exist models of arbitrarily large depth k which cannot be diminished by procedures of index d but can be decreased to d=const by procedures of index $d + 1$.

(3) It follows from Theorem 3 that for a set A, there does not exist a finite complete system of equivalent transforms for the models of sequential choice on A.

References

[1] M.A. Aizerman, F.T. Aleskerov, Choice Among Alternatives: Fundamentals of the Theory, Nauka, Moscow, 1990. (in Russian)

[2] M.A. Aizerman, A.V. Malishevski, Some aspects of the general theory of choosing the best alternatives, Avtomat, Telemekh. 2 (1981) 65–83 (in Russian).

[3] M.A. Aizerman, A.V. Malishevski, Conditions for universal reducibility of a two-stage extremization problem to one-stage problem, J. Math. Anal. Appl. 119 (1–2) (1986) 361–388.

[4] M.R. Garey, D.S. Johnson, Computers and Intractability—A Guide to the Theory of NP-completeness, Freeman, San Francisco, 1979.

[5] A.V. Malishevski, Preserving rationality in two-stage models of optimal choice, in: Data Analysis and Expert Evaluation in Organizing Systems, Inst. probl. upravl., (Moscow, 1985) 51–61 (in Russian).

[6] L.A. Sholomov, Logical Methods for Investigation of Discrete Choice Models, Nauka, Moscow, 1989. (in Russian)

[7] V.N. Vapnik, A.Ya. Chervonenkis, Pattern Recognition Theory: Statistical Problems of Learning, Nauka, Moscow, 1974. (in Russian)

[8] V.I. Vol'sky, Characteristic conditions for a class of choice functions, Systems and Control Lett. 2 (3) (1982) 189–193.
[9] V.I. Vol'sky, Application of two-stage procedures of multicriterial optimization to computer-aided process control systems, Preprints of the IFAC/IFORS Conference of Control Science and Technology for Development, Vol. 1, Beijing, China, 1985, pp. 467–476.
[10] Yu.I. Zhuravlev, Local algorithms for computing information I, Kibernetika 1 (1965) 12–19 (in Russian).
[11] Yu.I. Zhuravlev, Local algorithms for computing information II, Kibernetika 3 (1966) 1–11 (in Russian).
[12] Yu.I. Zhuravlev, G.F. Losev, Neighbourhoods in problems of discrete mathematics, Kibernetika Sistemnyi Anal. 2 (1995) 32–41 (in Russian).